Adaptive state space models
with applications to the business cycle and financial stress

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TVP - State space model

$$
\begin{aligned}
y_t &= Z_t \alpha_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, H_t) \\
\alpha_{t+1} &= T_t \alpha_t + \eta_t, \quad \eta_t \sim \mathcal{N}(0, Q_t), \quad t = 1, ..., n
\end{aligned}
$$

- y_t is the $N \times 1$ vector of obs. vars
- α_t is the $m \times 1$ state vector
- Z_t, H_t, T_t and Q_t are TV system matrices

- the model is conditionally Gaussian

$$
y_t | Y_{t-1}; \theta \sim N(Z_t a_t, F_t), \quad \alpha_t | Y_{t-1}; \theta \sim N(a_t, P_t),
$$

$$
\ell_t(y_t | Y_{t-1}; \theta) \propto -\frac{1}{2}\left(\log |F_t| + v_t' F_t^{-1} v_t\right),
$$

a_t, P_t, v_t, F_t are recursively computed by the KF

Assume that the model is score driven

Stack all the TVP in a vector $f_t = vec(Z_t, H_t, T_t, Q_t)$

$$
\begin{aligned}
f_{t+1} &= f_t + B s_t, \\
s_t &= \mathcal{S}_t^{-1} \nabla_t, \\
\nabla_t &= \frac{\partial \ell_t}{\partial f_t}, \\
\mathcal{S}_t &= -E_t \left(\frac{\partial \ell_t^2}{\partial f_t \partial f_t'} \right) = \mathcal{I}_t
\end{aligned}
$$

Why the score?

- Maximizes the local fit at t
- A variety of models, such as GARCH, DCC, MEM, Copula, can be cast in this framework; see Creal et al (2013)
- The score inherits the properties of the distribution of y_t
- They can be estimated by ML:

$$\widehat{\theta} = \arg\max \sum_{t=1}^{n} \ell_t(y_t | f_t, Y_{t-1}, \theta),$$

A new general algorithm for TVP-SS models

- KF at time t: given f_t, a_t and P_t

$$v_t = y_t - Z_t a_t, \quad F_t = Z_t P_t Z_t' + H_t, \quad K_t = T_t P_t Z_t' F_t^{-1},$$
$$a_{t+1} = T_t a_t + K_t v_t, \quad P_{t+1} = T_t P_t T_t' - K_t F_t K_t' + Q_t,$$
$$\ell_t(y_t | f_t, Y_{t-1}; \theta) \propto -\frac{1}{2}\left(\log |F_t| + v_t' F_t^{-1} v_t\right)$$

- KF at time $t+1$

$$v_{t+1} = y_{t+1} - Z_{t+1} a_{t+1}, \quad F_{t+1} = Z_{t+1} P_{t+1} Z_{t+1}{}' + H_{t+1}$$

- Now, to compute Z_{t+1} and H_{t+1} we need to know

$$s_t = \mathcal{I}_t^{-1} \nabla_t$$

The (scaled) score vector

$$\nabla_t = \frac{1}{2}\left[\overset{\bullet}{F}{}'_t(F_t^{-1} \otimes F_t^{-1})[v_t \otimes v_t - vec(F_t)] - 2\overset{\bullet}{V}{}'_t F_t^{-1} v_t\right]$$

$$\mathcal{I}_t = \frac{1}{2}\left[\overset{\bullet}{F}{}'_t(F_t^{-1} \otimes F_t^{-1})\overset{\bullet}{F}_t + 2\overset{\bullet}{V}{}'_t F_t^{-1} \overset{\bullet}{V}_t\right]$$

- $\overset{\bullet}{V}_t = \partial v_t / \partial f'_t$
- $\overset{\bullet}{F}_t = \partial vec(F_t) / \partial f'_t$
- How do we compute $\overset{\bullet}{V}_t$ and $\overset{\bullet}{F}_t$?

The additional recursions

$$
\begin{aligned}
\dot{V}_t &= -[(a_t' \otimes I_N)\dot{Z}_t + Z_t\dot{A}_t], \\
\dot{F}_t &= 2N_N(Z_tP_t \otimes I_N)\dot{Z}_t + (Z_t \otimes Z_t)\dot{P}_t + \dot{H}_t, \\
\dot{K}_t &= (F_t^{-1}Z_tP_t \otimes I_m)\dot{T}_t + (F_t^{-1}Z_t \otimes T_t)\dot{P}_t \\
&\quad + (F_t^{-1} \otimes T_tP_t)C_{Nm}\dot{Z}_t - (F_t^{-1} \otimes K_t)\dot{F}_t, \\
\dot{A}_{t+1} &= (a_t' \otimes I_m)\dot{T}_t + T_t\dot{A}_t + (v_t' \otimes I_m)\dot{K}_t + K_t\dot{V}_t, \\
\dot{P}_{t+1} &= (T_t \otimes T_t)\dot{P}_t - (K_t \otimes K_t)\dot{F}_t + \dot{Q}_t \\
&\quad + 2N_m[(T_tP_t \otimes I_m)\dot{T}_t - (K_tF_t \otimes I_m)\dot{K}_t].
\end{aligned}
$$

for a generic matrix M_t, we have that $\dot{M}_t = \partial vec(M_t)/\partial f_t'$

Summary

Algorithm

Initialize the following elements a_1, P_1, f_1, $\dot{A}_1$, $\dot{P}_1$.

For $t = 1, ..., n$:

i. compute v_t, F_t, K_t, and ℓ_t

ii. compute $\dot{V}_t$, $\dot{F}_t$, $\dot{K}_t$

iii. compute ∇_t, $\mathcal{I}_t$, (and s_t)

iv. update a_{t+1}, P_{t+1}

v. update $\dot{A}_{t+1}$, $\dot{P}_{t+1}$

vi. update f_{t+1}

Other things we discuss in the paper

- Impose parameter restrictions (for instance positive variances)
- Handle missing obs and mixed frequency data
- How the method generalizes other approaches (for instance VAR models with TVP and forgetting factors)

Monte Carlo: DGPs for the observations y_t

- Time-Varying loadings in the measurement:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} 1 \\ f_t \end{bmatrix} \mu_t + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}, \quad \varepsilon_t \sim \mathcal{N}(0, I)$$
$$\mu_t = 0.8\mu_{t-1} + u_t \qquad u_t \sim \mathcal{N}(0, 1)$$

- Time-Varying AR coefficient in the transition:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \mu_t + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}, \quad \varepsilon_t \sim \mathcal{N}(0, I)$$
$$\mu_t = f_t\mu_{t-1} + u_t \qquad u_t \sim \mathcal{N}(0, 1)$$

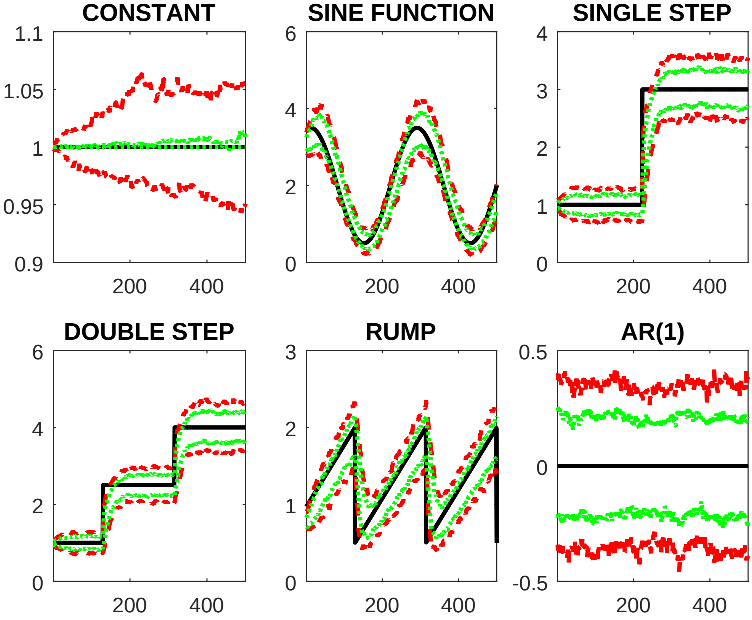
- Time-Varying Volatilities:

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}\left(0, \sigma^2_{\varepsilon,t}\right), \quad \varepsilon_t \sim \mathcal{N}(0, 1), \\ \mu_{t+1} &= 0.8\mu_t + u_t, \quad u_t \sim \mathcal{N}(0, 1), \quad u_t \sim \mathcal{N}\left(0, \sigma^2_{\eta,t}\right) \end{aligned}$$

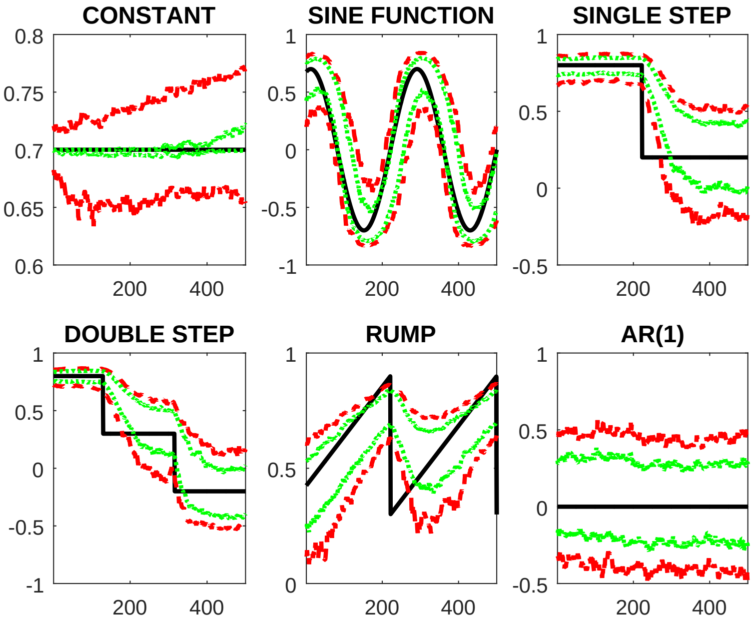
Monte Carlo results

	T = 250						T = 500					
	RMSE	MAE	Corr.	68%	90%	#Pile	RMSE	MAE	Corr.	68%	90%	#Pile
				TIME-VARYING LOADING								
CONSTANT	0.000	0.000		0.680	0.896	215	0.00	0.000		0.682	0.898	223
SINE	0.493	0.396	0.899	0.640	0.860	0	0.390	0.311	0.939	0.653	0.870	0
SINGLE STEP	0.433	0.294	0.915	0.656	0.872	0	0.362	0.236	0.942	0.660	0.882	0
DOUBLE STEP	0.485	0.348	0.930	0.648	0.868	0	0.390	0.276	0.952	0.652	0.874	0
RAMP	0.431	0.322	0.463	0.680	0.892	0	0.350	0.247	0.641	0.680	0.892	0
AR(1)	0.207	0.169	0.649	0.672	0.892	19	0.213	0.171	0.727	0.680	0.898	2
				TIME-VARYING AR COEFF								
CONSTANT	0.000	0.000		0.676	0.896	197	0.000	0.000		0.680	0.899	204
SINE	0.361	0.291	0.781	0.676	0.896	0	0.281	0.219	0.870	0.682	0.900	0
SINGLE STEP	0.269	0.193	0.718	0.676	0.896	0	0.225	0.154	0.783	0.678	0.897	0
DOUBLE STEP	0.279	0.220	0.845	0.684	0.900	0	0.235	0.175	0.880	0.682	0.898	0
RAMP	0.193	0.153	0.190	0.680	0.896	0	0.178	0.135	0.379	0.680	0.896	0
AR(1)	0.243	0.195	0.537	0.678	0.896	48	0.252	0.204	0.634	0.682	0.900	15
				TIME-VARYING VOLATILITY - MEAS. ERROR								
CONSTANT	0.00	0.00		0.674	0.892	229	0.00 0	0.000		0.678	0.898	249
SINE	0.582	0.466	0.551	0.688	0.896	24	0.444	0.360	0.725	0.691	0.896	0
SINGLE STEP	0.493	0.359	0.843	0.636	0.856	0	0.382	0.272	0.881	0.652	0.872	0
DOUBLE STEP	0.448	0.355	0.843	0.624	0.844	1	0.355	0.272	0.881	0.650	0.868	0
RAMP	0.603	0.501	0.200	0.648	0.868	163	0.514	0.417	0.368	0.662	0.878	30
AR(1)	0.341	0.281	0.476	0.676	0.892	300	0.341	0.274	0.513	0.678	0.892	159
				TIME-VARYING VOLATILITY - TRANS. ERROR								
CONSTANT	0.000	0.000		0.680	0.896	249	0.000	0.000		0.680	0.898	247
SINE	0.571	0.468	0.578	0.680	0.896	105	0.503	0.405	0.702	0.681	0.896	14
SINGLE STEP	0.508	0.374	0.813	0.652	0.876	1	0.421	0.296	0.859	0.664	0.884	0
DOUBLE STEP	0.472	0.375	0.828	0.644	0.866	0	0.369	0.283	0.878	0.660	0.880	0
RAMP	0.616	0.522	0.162	0.656	0.876	39	0.535	0.442	0.319	0.666	0.886	7
AR(1)	0.375	0.305	0.482	0.672	0.892	318	0.352	0.283	0.532	0.674	0.894	188

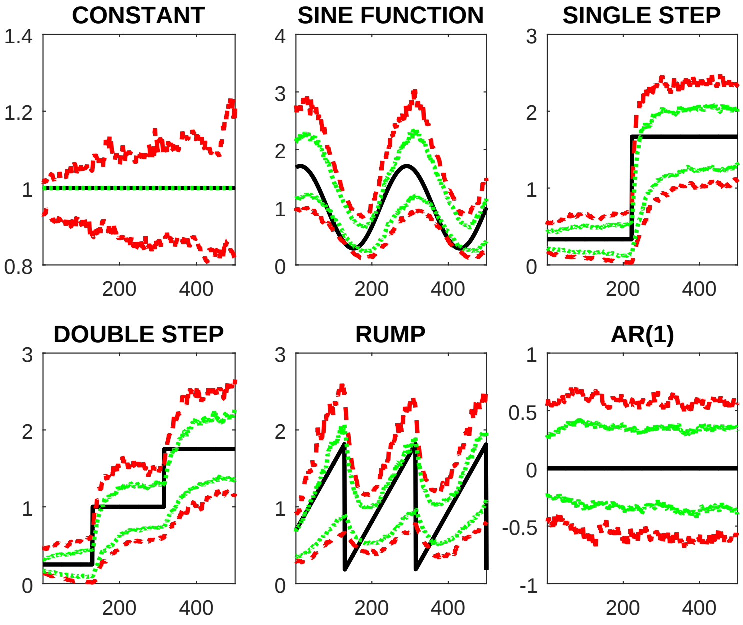
Monte Carlo results: time-varying loadings



Monte Carlo results: time-varying AR



Monte Carlo results: time-varying volatility



Financial Stress and Business Cycle

- We build a measure of financial stress in the euro area by combining data on:
  - Financial indicators
  - Business cycle indicators
- Twofold aim
  1. Explore time variation in the link between the financial and the business cycle
  2. Check whether financial data are useful for *nowcasting* GDP
- Nice illustration of the flexibility of our method
  1. Quarterly and monthly data
  2. Data starting in different periods
  3. Ragged edges

Data

- Indicators of economic activity
  - GDP
  - IP
  - Surveys (PMI, composite and orders)

- Financial indicators
  - Money Market: RV and spreads of short-term rates+Emergency Central Bank lending
  - Bond Market: RV and spreads of long-term rates
  - Stock Market: RV stock market indexes, Stock-bond correlation
  - Banking sector: RV and spreads financial non financial
  - Foreign Exhange Market: RV of the euro exchange rate vis-à-vis the US dollar, Japanese Yen, the British Pound

Model specification

The measurement equation is block diagonal

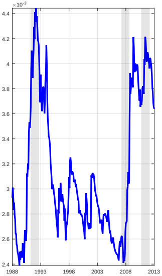
$$\begin{bmatrix} y_t^r \\ y_t^x \end{bmatrix} = \begin{bmatrix} \lambda_t^r & 0 \\ 0 & \lambda_t^x \end{bmatrix} \begin{bmatrix} \alpha_t^r \\ \alpha_t^x \end{bmatrix} + \begin{bmatrix} \varepsilon_t^r \\ \varepsilon_t^x \end{bmatrix}, \qquad \begin{bmatrix} \varepsilon_t^r \\ \varepsilon_t^x \end{bmatrix} \sim \mathcal{N}(0, H_t).$$

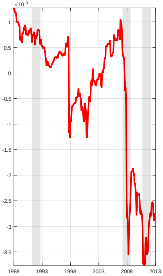
The transition equation imposes Granger-causality restrictions

$$\begin{bmatrix} \alpha_{t+1}^r \\ \alpha_{t+1}^x \end{bmatrix} = \begin{bmatrix} \phi_{11,t} & \phi_{12,t} \\ 0 & \phi_{22,t} \end{bmatrix} \begin{bmatrix} \alpha_{t-1}^r \\ \alpha_{t-1}^x \end{bmatrix} + u_t, \qquad u_t \sim \mathcal{N}(0, \Sigma_t)$$

Time varying covariances

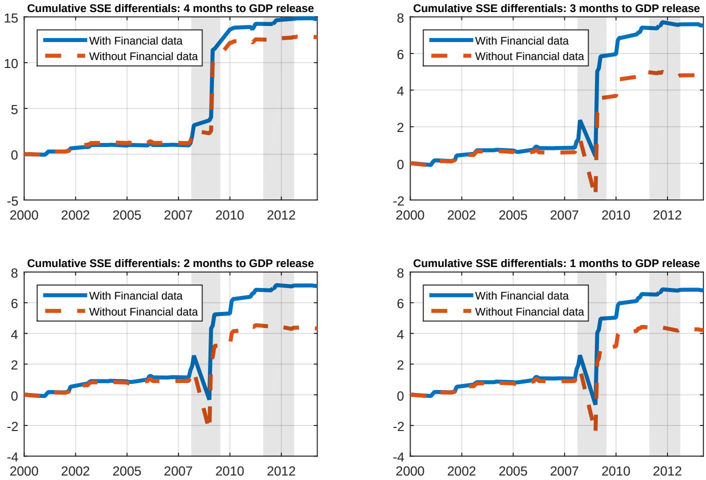
- BusCycle Volatility rises in recessions
- FinStress Volatility rises after 2008
- Co-variation of BusCycle and FinStress is zero in non–crisis periods and strongly increases (in abs value) after 2008





Cumulative Sum of Squared Errors Differentials

- Most of the gains are episodic to the Great Financial Crisis



Density forecast evaluation

- Yet financial data help in calibrating well density forecasts
- Especially at the left–hand tail (vulnerability)